SPATIAL LOCALIZATION OF THERMAL PERTURBATIONS IN HEATING A MEDIUM WITH BULK HEAT ABSORPTION

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UDC 517.946

Solutions are constructed of a type of thermal wave describing stationary and nonstationary heating processes of a medium with constant thermal conducting and with bulk heat absorption.

For isotropic media, whose thermal conductivity coefficient $\varkappa$ is a power of temperature T, $\varkappa = \varkappa_0 T^k$, $\varkappa_0$, k = const > 0, thermal perturbations propagate from the heat source with constant velocity [1]. If there is bulk heat absorption in the medium, motion of the thermal heat front, separating regions with $\nabla T = 0$ and $\nabla T \neq 0$, can occur only at a definite finite distance [2].

The spatial localization of thermal perturbation and the restricted advance of the thermal wave front can be also observed in case of media with a constant thermal conductivity coefficient $\varkappa_0$ in the presence of bulk heat absorption in them, "thermal sinks." We consider below the heating process of an isotropic medium with $\varkappa = \varkappa_0$, filling the half-space z > 0 and having a constant initial temperature T_0 = const > 0, when the surface temperature at z = 0, starting at time t = 0, varies by the law $T = T_W(t)$.

If thermal sinks f act in a medium with a constant thermal conductivity, and we assume the existence of a thermal wave front $z = \zeta(t)$, the temperature distribution T(z, t) and $0 \leq z \leq \zeta(t)$ is determined from the solution of the problem

$$\begin{gathered}\partial T / \partial t = a\partial^2 T / \partial z^2 - f(z, t, T), \quad T(z, 0) = T_0 \\ T(0, t) = T_w(t), \quad T[\zeta(t), t] = T_0, \quad \frac{\partial T}{\partial z}[\zeta(t), t] = 0\end{gathered} \tag{1}$$

Here and in what follows $a = \varkappa_0 c^{-1}\rho^{-1}$ is the constant thermal conductivity coefficient, and c and ρ are the specific heat and density of the medium. In the open region D of the phase plane zt ($0 < z < \zeta(t)$, $0 < t < \tau < \infty$) it is required to determine a function T(z, t), continuous together with its derivative $\partial T(z, t)/\partial z$ everywhere in the closed region $\bar{D}$, except, perhaps, the point (0,0). The solution of problem (1) assumes a definite law of motion of the unknown boundary $z = \zeta(t)$, the thermal wave front. For $\zeta(t) < z < \infty$, $0 < t < \tau < \infty$ the temperature distribution is $T(z, t) = T_0$.

The existence of regions with different analytic expressions for the temperature distribution, characteristic of solutions of the thermal wave type, is ultimately related to singular solutions of ordinary differential equations, while at the same time there is no difference between media with a constant thermal conductivity and media with a temperature-dependent thermal conductivity.

To explain the analytic nature of solutions of the thermal wave type, consider the problem of determining the stationary temperature distribution in a medium with a constant thermal conductivity and thermal sinks of the form

$$\begin{gathered}f = \gamma T^n \theta(T - T_0), \quad T \geqslant T_0 \\ \gamma = \text{const} > 0, \; n = \text{const} > -1, \; \theta(T - T_0) = \lim_{N \to \infty}(T - T_0)^{1/N}\end{gathered} \tag{2}$$

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 96-101, September-October, 1973. Original article submitted April 18, 1973.

with a constant temperature value $T_m > T_0$, at the surface $z = 0$. Thermal waves, found by solving (1), can be obtained from the problem

$$\begin{gathered}\partial T/\partial t = a\partial^2 T/\partial z^2 - f(z, t, T), \quad T(z, 0) = T_0 \\ T(0, t) = T_w(t), \quad T(\infty, t) = T_0, \quad \frac{\partial T}{\partial z}(\infty, t) = 0 \\ T[\zeta(t) - 0, t] = T[\zeta(t) + 0, t], \quad \frac{\partial T}{\partial z}[\zeta(t) - 0, t] = \frac{\partial T}{\partial z}[\zeta(t) + 0, t]\end{gathered} \tag{3}$$

where $0 < z < \infty$, $0 < t < \tau < \infty$. The identity of solutions of problems (1) and (3) for the same thermal sinks f can be established if both solutions are represented in integral form by means of the source function for a semi-infinite straight line [3]. Integrating once the stationary equation (3) with boundary conditions (3) for $z \to \infty$, one obtains

$$dT/dz = A(T^{n+1} - T_0^{n+1})^{1/2}, \quad A \equiv -[2\gamma/a(1+n)]^{1/2} \tag{4}$$

Obviously, Eq. (4) is satisfied by the solution $T = T_0$, and the partial solution arriving at the zT plane through the point $(0, T_{m0})$ is described in the form

$$Az = \int\limits_{T_{m0}}^{T} \frac{dT}{(T^{n+1} - T_0^{n+1})^{1/2}} \tag{5}$$

For $T \to T_0$ the integral in (5) becomes improper. It converges if $T_0 \geq 0$ for $-1 < n < 1$, or if $T_0 > 0$ for $n \geq 1$. In these cases $z \to \zeta_0 < \infty$, i.e., the integral curve (5) passes the zT plane at the point (ζ_0, T_0) on the straight line $T = T_0$. Consequently, the solution $T = T_0$ is a singular solution, since uniqueness [4] is violated in all its points. On the zT plane the singular solution $T = T_0$ is the envelope of the set of partial solutions of Eq. (4). For exactly this reason the stationary solution of problem (2), (3) can be represented as a function, joined at the point $z = \zeta_0$ from the partial solution (5), which determines the temperature distribution in the perturbed region at $0 \leq z \leq \zeta_0$ and the singular solution $T = T_0$, which determines the constant temperature in the unperturbed region at $\zeta_0 \leq z < \infty$.

Analyzing the model problem describing nonstationary heating processes of media considered in [1, 2, 5] and in the present paper, it can be verified that the ordinary differential equation which should be integrated to solve such problems has a singular solution. The existence of a singular solution guarantees the possibility of joining the integral surfaces $T(z, t) \neq$ const and $T(z, t) = T_0 =$ const along the line $z = \zeta(t)$, describing the law of motion of a thermal wave front.

The connection between solutions of the thermal wave front and the existence of singular solutions of the corresponding ordinary differential equations can also be established for arbitrary nonstationary regions of heated media. Indeed, for sufficiently short time intervals during which the propagation velocity of a thermal wave can be assumed constant, the temperature distribution close to the front is described in a coordinate attached to the wave front by an ordinary differential equation with a singular solution. The singular solution corresponds always to a constant temperature in the unperturbed portion of the medium, while the temperature distribution in the perturbed region is described by the partial solution, joined to the singular solution at the surface of the thermal wave front.

We turn now to concrete examples of solutions of the thermal wave type describing some heating region of a medium with a constant thermal conductivity and bulk heat absorption.

Let

$$\begin{gathered}f = f_1 \equiv \gamma_1 \exp(\alpha[z - \zeta(t)])\,\theta(T - T_0) \\ \alpha,\ T_0 = \text{const} \geqslant 0, \quad \gamma_1 = \text{const} > 0\end{gathered} \tag{6}$$

If the wall ($z = 0$) temperature increases monotonically according to the expression

$$T_w(t) = T_0 + \frac{\gamma_1}{v\alpha}\left[\frac{a\alpha \exp(v^2 t/a) + v\exp(-\alpha v t)}{v + a\alpha} - 1\right], \quad v = \text{const} > 0 \tag{7}$$

the temperature distribution in the medium can be determined by solving the problem (1), (6), (7). As a result one obtains

$$T(z,t)=\begin{cases} T_0+\frac{\gamma_1}{v\alpha}\left[\frac{a\alpha\exp[-v(z-vt)/q]+v\exp[\alpha(z-vt)]}{v+a\alpha}-1\right] & \text{for } 0\leqslant z\leqslant\zeta(t)\\ T_0 & \text{for } \zeta(t)\leqslant z<\infty \end{cases}\tag{8}$$
$$\zeta(t)=vt$$

There exists, thus, in the medium a thermal wave front $z=\zeta(t)$, moving with constant velocity from the surface $z=0$.

We note that the problem (1), (6) has a stationary solution if the surface $z=0$ is subject to a constant temperature $T_W(t)=T_{m0}=\text{const}$, $T_{m0}>T_0$. In this case the temperature distribution in the medium is determined by the expressions

$$T(z)=\begin{cases} T_0+\frac{\gamma_1}{a\alpha^2}[\exp[\alpha(z-\zeta_0)]-\alpha(z-\zeta_0)-1] & \text{for } 0\leqslant z\leqslant\zeta_0\\ T_0 & \text{for } \zeta_0\leqslant z<\infty \end{cases}\tag{9}$$

while the location of the immobile thermal wave front $z=\zeta_0$, i.e., the boundary of the warmed layer of the medium flowing to the heated surface, should be found by solving the transcendental equation

$$\exp(-\alpha\zeta_0)+\alpha\zeta_0=1+(a\alpha^2/\gamma_1)(T_{m0}-T_0)\tag{10}$$

We determine the temperature distribution for an oscillating region of the surface ($z=0$) temperature

$$T_w(t)=T_{m0}+T_{m1}e^{i\omega t},\quad T_{m0},\ T_{m1},\ \omega=\text{const}>0,\qquad T_{m0}-T_0>T_{m1}\tag{11}$$

For simplicity we assume that in the expression of thermal sinks (9) $\alpha\equiv 0$, so that the solution of the problem without initial conditions (1), (6), (11) is reasonably represented in the form

$$T(z,t)=\begin{cases} T_{m0}+T_{m1}\operatorname{ch}\Omega_1 z\exp(i\omega t)+\frac{\gamma_1 z^2}{2a}+A_0 z+\sum\limits_{l=0}^{\infty}A_l\operatorname{sh}\Omega_l z\exp(il\omega t) & \text{for } 0\leqslant z\leqslant\zeta(t)\\ T_0 & \text{for } \zeta(t)\leqslant z<\infty \end{cases}\tag{12}$$

$$\zeta(t)=\sum_{j=0}^{\infty}\zeta_j\exp(ij\omega t),\quad \Omega_l=(1+i)\sqrt{\omega l/2a},\quad A_0,\ A_1,\ldots,\ \zeta_0,\ \zeta_1,\ldots-\text{const}$$

The function $T(z,t)$ of (12) satisfies the differential equation and boundary conditions (1) at the surface $z=0$ with $T_W(t)$ determined according to (11). From the expressions for $T(z,t)$ and $\zeta(t)$ (12) and the boundary conditions (1) at the thermal wave front $z=\zeta(t)$ one can find the constants $A_0, A_1, \ldots, \zeta_0, \zeta_1\ldots$

$$\begin{aligned} \zeta_0&=\sqrt{\frac{2a}{\gamma_1}(T_{m0}-T_0)},\quad \zeta_1=\frac{aT_{m1}\Omega_1}{\gamma_1\operatorname{sh}\Omega_1\zeta_0},\\ \zeta_2&=-\frac{a^2T_{m1}^2\Omega_1^2\Omega_2\operatorname{cth}\Omega_2\zeta_0}{2\gamma_1\operatorname{sh}^2\Omega_1\zeta_0}\ldots\\ A_0&=-\sqrt{\frac{2\gamma_1}{a}(T_{m0}-T_0)},\quad A_1=-T_{m1}\operatorname{cth}\Omega_1\zeta_0,\\ A_2&=\frac{aT_{m1}^2\Omega_1^2}{2\gamma_1\operatorname{sh}^2\Omega_1\zeta_0\operatorname{sh}\Omega_2\zeta_0}\ldots \end{aligned}\tag{13}$$

We note that the expression ζ_0 (13) determines the stationary location of the thermal wave front in the medium if the surface $z=0$ is subject to a constant temperature $T_{m0}>T_0>0$.

We assume now that thermal sinks of the form

$$f=f_2\equiv {}^1/_2\,\gamma_2 t^{-1/2}\,\theta(T),\qquad \gamma_2=\text{const}>0\tag{14}$$

act in the medium.

If the surface ($z=0$) temperature varies by the law

$$T_w(t)=\gamma_2[\exp(\xi_0^2/2)-1]\sqrt{t},\ \ \xi_0=\text{const}>0\tag{15}$$

and the initial medium temperature is $T_0 = 0$, the temperature distribution in the medium can be determined by solving the model problem (1), (14), (15). As a result we obtain

$$T(z,t) = \begin{cases} \gamma_2 \sqrt{t} \left\{ \exp\left[(\xi_0^2 - \xi^2)/2\right] + \frac{\sqrt{\pi}}{2} \xi \exp(\xi_0/2) \left[\Phi(\xi) - \Phi(\xi_0)\right] - 1 \right\} & \text{for} \quad 0 \leqslant z \leqslant \zeta(t) \\ 0 & \text{for} \quad \zeta(t) \leqslant z < \infty \end{cases} \tag{16}$$

$$\zeta(t) = \xi_0 \sqrt{2at}, \quad \xi = z/\sqrt{2at}$$

where $\Phi(\xi)$ and $\Phi(\xi_0)$ are error integrals.

Consider thermal sinks of the form

$$f = f_3 \equiv \gamma_3 T \theta (T - T_0), \qquad \gamma_3,\ T_0 = \text{const} > 0 \tag{17}$$

We note that it is possible, in principle, to consider the more general action law of thermal sinks f_3, contained in the first part of Eq. (17) in the exponential factor $\exp(\alpha[z-\zeta(t)])$. The corresponding results, however, are not given here due to the awkwardness of the final relations.

If the surface ($z = 0$) temperature oscillates harmonically by (11), the temperature distribution in the medium can be determined by solving the problem without initial conditions (1), (11), (17). Corresponding calculations lead to

$$T(z,t) = \begin{cases} T_{m0} \operatorname{ch} \Omega_0 z + T_{m1} \operatorname{ch} \Omega_1 z \exp i\omega t + \sum\limits_{l=0}^{\infty} A_l \operatorname{sh} \Omega_l z \exp il\omega t & \text{for} \quad 0 \leqslant z \leqslant \zeta(t) \\ T_0 & \text{for} \quad \zeta(t) \leqslant z < \infty \end{cases} \tag{18}$$

$$\zeta(t) = \sum_{j=0}^{\infty} \zeta_j \exp ij\omega t$$

$$\Omega_l = \left(\frac{il\omega + \gamma_3}{a}\right)^{1/2}, \quad \zeta_0 = \frac{1}{\Omega_0} Ar \operatorname{ch} \frac{T_{m0}}{T_0}, \quad \zeta_1 = \frac{T_{m1}\Omega_1}{T_0 \Omega_0^2 \operatorname{sh} \Omega_1 \zeta_0}$$

$$\zeta_2 = -\frac{T_{m_1}^2 \Omega_1^2 \Omega_2 \operatorname{cth} \Omega_2 \zeta_0}{2T_0^2 \Omega_0^4 \operatorname{sh}^2 \Omega_1 \zeta_0} \ldots, \quad A_0 = -\sqrt{T_{m0}^2 - T_0^2}$$

$$A_1 = -T_{m1} \operatorname{cth} \Omega_1 \zeta_0, \quad A_2 = \frac{T_{m1}^2 \Omega_1^2}{2T_0 \Omega_0^2 \operatorname{sh} \Omega_2 \zeta_0 \operatorname{sh}^2 \Omega_1 \zeta_0} \ldots \tag{19}$$

The expression of ζ_0 (19) determines the stationary location of the thermal wave front in the medium if the surface $z = 0$ is subject to a constant temperature $T_{m0} > T_0 > 0$.

If the surface ($z = 0$) temperature varies according to

$$T_w(t) = T_0 e^{\beta v t} (\operatorname{ch} \delta vt - \beta \delta^{-1} \operatorname{sh} \delta vt), \ v = \text{const} > 0$$

$$\beta = v/2a, \ \delta = \sqrt{v^2/4a^2 + \gamma_3/a} \quad , \tag{20}$$

the temperature distribution in the medium, obtained by solving the model problem (1), (17), (20), has the following form:

$$T(z,t) = \begin{cases} T_0 \exp\left[-\beta(z - vt)\right] \left[\operatorname{ch} \delta (z - vt) + \frac{\beta}{\delta} \operatorname{sh} \delta (z - vt)\right] & \text{for} \quad 0 \leqslant z \leqslant \zeta(t) \\ T_0 & \text{for} \quad \zeta(t) \leqslant z < \infty \end{cases} \tag{21}$$

$$\zeta(t) = vt$$

i.e., the thermal wave front $z = \zeta(t)$ moves in the medium with constant velocity v. We assume that thermal sinks of the form

$$f = f_4 \equiv \gamma_4 T^\beta \theta(T), \quad \beta, \gamma_4 = \text{const}, \ |\beta| < 1, \quad \gamma_4 > 0 \quad . \tag{22}$$

act in the medium with $T_0 \equiv 0$.

When the surface ($z = 0$) temperature is $T_W(t) = T_{m0} = \text{const} > 0$, the following temperature distribution [2] occurs in the medium:

$$T(z) = \begin{cases} T_{m0}(1 - z/\zeta_0)^{2/(1-\beta)} & \text{for} \quad 0 \leqslant z \leqslant \zeta_0 \\ 0 & \text{for} \quad \zeta_0 \leqslant z < \infty \end{cases}$$

$$\zeta_0 = [2a(1+\beta)/\gamma_4 T_{m0}^{\beta-1}(1-\beta)^2]^{1/2} \tag{23}$$

If the surface ($z=0$) temperature in the initial moment of time $t=0$ varies discontinuously from $T_0=0$ to $T_W(t) = T_{m0} =$ const, taking into account the stationary solution (23), an approximate expression of the temperature distribution can be sought in form

$$T(z, t) = T_{m0}[1 - z/\zeta(t)]^{2/(1-\beta)} . \tag{24}$$

Since

$$T[\zeta(t), t] = \frac{\partial T}{\partial z}[\zeta(t), t] = 0 \ ,$$

by integrating the differential equation (1) over z from 0 to $\zeta(t)$ one obtains an integral equation of thermal balance

$$\frac{d}{dt}\int_0^{\zeta(t)} T(z,t)\,dz = -a\frac{\partial T}{\partial z}(0,t) - \gamma_4\int_0^{\zeta(t)} T^{\beta}(z,t)\,dz \tag{25}$$

Substituting (24) in (25), one can obtain an approximate law of motion of the thermal wave front $z = \zeta(t)$, satisfying the condition $\zeta(0) = 0$

$$\zeta(t) = \zeta_0\left\{1 - \exp\left[-\frac{4\gamma_4 T_{m0}^{\beta-1}(1-\beta)}{1+\beta}\,t\right]\right\}^{1/2} \tag{26}$$

where ζ_0 is determined by (23). Within the same approximation one can estimate the relaxation time to the stationary region (23)

$$\tau_R \approx T_{m0}^{1-\beta}(1+\beta)/4\gamma_4(1-\beta) \tag{27}$$

The examples given indicate, indeed, that spatially localized thermal waves whose front propagates with finite velocity from the source of thermal perturbations can be formed in media with bulk heat absorption. The necessity of studying the solution of the thermal conductivity equation with a term describing sinks f arises, in particular, in considering the process of heat propagation in a thin rod of layer, accompanied by heat transfer to surrounding space. We also note that the solutions mentioned can be successfully applied to other transport processes, such as gas diffusion in simple media with high absorption.

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